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DEGENERATE CAUCHY NUMBERS AND POLYNOMIALS OF THE FOURTH KIND

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Abstract. In this paper, we define degenerate Cauchy numbers of the forth kind. We provide an explicit formula for each the degenerate Cauchy polynomials and numbers of the fourth kind. In addition, we obtain some identities for these polynomials and numbers.

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1. Introduction

Throughout this article, $\mathbb{N}^*$ and $\mathbb{R}$ stand for the set of all nonnegative integers and the set of all real numbers respectively. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$, the symbol $(x)_n$ denotes the falling factorials, that is,

$$(x)_0 = 1,\ (x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

For $m, n \in \mathbb{N}$, we denote the Stirling numbers of the first kind and the second kind by $S_1(m,n)$ and $S_2(m,n)$ respectively. Each of them is defined by the generating function as follows:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k$$
$$x^n = \sum_{m=0}^{n} S_2(n,m)(x)_m.$$

It is well known that the *Cauchy polynomials*, denoted by $C_n(x)$, are defined by the following generating function :

$$\int_0^1 (1+t)^{x+y}dy = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!} \text{(see [13, 18, 19, 21])}. \tag{1}$$

When $x = 0$, $C_n(0) = C_n$ are called the *Cauchy numbers(or Bernoulli numbers of the second kind)*. It is well known that $C_n = \int_0^1 (x)_n dx$.

In [8] and [17], the Cauchy numbers are pointed out that it is very important for the study of mathematical physics. Various characteristics of Cauchy numbers can be found in [3, 7, 11, 22].

Since Calitz [2], the function $(1+\lambda t)^{\frac{1}{\lambda}}$ is called the degenerate function of e^t. So, for $t = \log e^t$, the equation $\log(1+\lambda t)^{\frac{1}{\lambda}}$ can be used as the degenerating function of t. A number of studies for some kinds degenerate functions have been conducted(see [4, 5, 6, 12, 9]).

In [14], T. Kim introduced the *degenerate Cauchy numbers* and *polynomials*, denoted by $C_{n,\lambda}$ and $C_{n,\lambda}(x)$, as follows:

(2)
$$\begin{aligned}\int_0^1 (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x+y}dy &= \frac{\frac{1}{\lambda}\log(1+\lambda t)}{\log(1+\frac{1}{\lambda}\log(1+\lambda t))}(1+\frac{1}{\lambda}\log(1+\lambda t))^x \\ &= \sum_{n=0}^{\infty} C_{n,\lambda}(x)\frac{t^n}{n!},\end{aligned}$$

where $x=0, C_{n,\lambda}=C_{n,\lambda}(0)$ are called the *degenerate Cauchy numbers*.

In addition, a lot of interesting identities are presented about the degenerate Cauchy numbers with other special numbers, the Cauchy numbers, Stirling numbers of the first kind and the second kind and higher order Beroulli numbers etc. One of them is the following.

$$C_{n,\lambda} = \sum_{l=0}^{\infty} \lambda^{n-l} S_1(n,l) C_l.$$

The *degenerate Cauchy polynomials of the second kind*, denoted by $C_{n,\lambda,2}(x)$, are defined by the generating function as follows [15]:

$$\frac{t}{\log(1+\frac{1}{\lambda}\log(1+\lambda t))}(1+\frac{1}{\lambda}\log(1+\lambda t))^x = \sum_{n=0}^{\infty} C_{n,\lambda,2}(x)\frac{t^n}{n!}. \tag{3}$$

In case $x=0$ in the equation (3), the numbers $C_{n,\lambda,2}=C_{n,\lambda,2}(0)$ are called the *degenerate Cauchy numbers of the second kind*.

In [15], Kim presented lots of identities by using the degenerate Cauchy numbers of the second kind. One of the interesting characteristics of the degenerate Cauchy numbers of the second kind is that it connects the degenerate Cauchy numbers to the Daehee numbers.

$$C_{n,\lambda} = \sum_{m=0}^{n} (n)_m \lambda^{n-m} D_{n-m} C_{m,\lambda,2}\frac{1}{m!}$$

where D_n are the Daehee numbers which are defined by the generating function to be

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}.$$

In [20], Pyo and et al. introduced the *degenerate Cauchy numbers of the third kind*, denoted by $C_{n,\lambda,3}$, which are defined by generating function

$$\begin{aligned}\int_0^1 (1+\lambda\log(1+t))^{\frac{x}{\lambda}}dx &= \frac{\lambda\left((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1\right)}{\log(1+\lambda\log(1+t))} \\ &= \sum_{n=0}^{\infty} C_{n,\lambda,3}\frac{t^n}{n!}.\end{aligned} \tag{4}$$

In [20], the degenerate Cauchy numbers of the third kind $C_{n,\lambda,3}$ are explicitly determined by the Stirling numbers of the first kind. Three identities, by using $C_{n,\lambda,3}$, about the Stirling numbers of the first kind and the Cauchy numbers are presented. In addition, four relations between the degenerate Cauchy numbers of third kind and other kinds of degenerate Cauchy numbers are presented.

Just as Kim introduced the degenerate Cauchy numbers of the second kind from the degenerate Cauchy numbers [15], Pyo and et al. defined the *degenerate Cauchy numbers of the fourth kind* as follows :

$$\frac{\lambda t}{\log(1+\lambda\log(1+t))} = \sum_{n=0}^{\infty} C_{n,\lambda,4}\frac{t^n}{n!}. \tag{5}$$

Note that each of all the limits of degenerate functions, the equations (2) $\sim$ (5), converge to the generating function of the Cauchy numbers and polynomials as λ goes to 0,

$$\begin{aligned} &\lim_{\lambda\to 0}\frac{\frac{1}{\lambda}\log(1+\lambda t)}{\log(1+\frac{1}{\lambda}\log(1+\lambda t))}(1+\frac{1}{\lambda}\log(1+\lambda t))^x \\ &= \lim_{\lambda\to 0}\frac{t}{\log(1+\frac{1}{\lambda}\log(1+\lambda t))}(1+\frac{1}{\lambda}\log(1+\lambda t))^x \\ &= \frac{t}{\log(1+t)}(1+t)^x, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\lim_{\lambda\to 0}\frac{\lambda\left((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1\right)}{\log(1+\lambda\log(1+t))} \\ &= \lim_{\lambda\to 0}\frac{\lambda t}{\log(1+\lambda\log(1+t))} \\ &= \frac{t}{\log(1+t)}. \end{aligned} \tag{7}$$

The equation (6) and (7) give us

$$\begin{aligned} \lim_{\lambda\to 0} C_{n,\lambda} &= \lim_{\lambda\to 0} C_{n,\lambda,2} = \lim_{\lambda\to 0} C_{n,\lambda,3} = \lim_{\lambda\to 0} C_{n,\lambda,4} \\ &= C_n. \end{aligned} \tag{8}$$

When $n=0$, we know that

$$C_0 = C_{0,\lambda} = C_{0,\lambda,2} = C_{0,\lambda,3} = C_{0,\lambda,4} = 1. \tag{9}$$

Figure 1. shows the four kinds of degenerate Cauchy numbers [20].

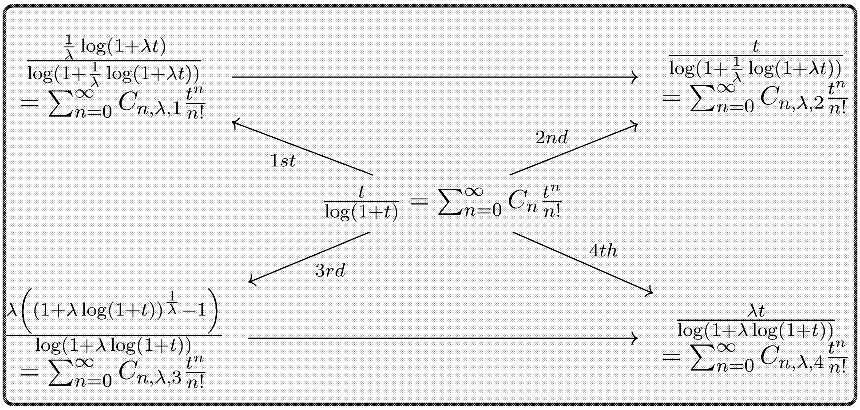

Figure 1. Four kinds of degenerate Cauchy numbers

In this paper, we provide an explicit formula for each degenerate Cauchy polynomials and numbers of the fourth kind. In addition, we obtain some identities for these polynomials and numbers.

2. Degenerate Cauchy polynomials of the fourth kind

Let us take a look at the next two variable function $F(x,t)$,

$$F(x,t) = \frac{\lambda t}{\log(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}. \tag{10}$$

If λ goes to 0 in the equation (10), then the function $F(x,t)$ becomes the generating function of Cauchy polynomials,

$$\lim_{\lambda\to 0}\frac{\lambda t}{\log(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}} = \frac{t}{\log(1+t)}(1+t)^x. \tag{11}$$

Like as the equation (6) and (7), the function $F(x,t)$ is a generating function of degenerate Cauchy polynomial type, we call this a generating function of the *degenerate Cauchy polynomials of the fourth kind* and represent it as follows:

$$\frac{\lambda t}{\log(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} C_{n,\lambda,4}(x)\frac{t^n}{n!}. \tag{12}$$

The the coefficients $C_{n,\lambda,4}(x)$ of $\frac{t^n}{n!}$ are called the *degenerate Cauchy polynomials of the fourth kind*, when $x=0$, $C_{n,\lambda,4} = C_{n,\lambda,4}(0)$ are called the degenerate Cauchy numbers of the fourth kind.

The equation (11) noted that $C_{n,\lambda,4}(x)$ converges to $C_n(x)$ as λ goes to 0.

When $x=0$, the equation (10) becomes the generating function of the degenerate Cauchy numbers of the fourth kind,

$$\frac{\lambda t}{\log(1+\lambda\log(1+t))} = \sum_{n=0}^{\infty} C_{n,\lambda,4}\frac{t^n}{n!}. \tag{13}$$

In [16], T. Kim introduced λ-*analogue falling factorials* and presented several results regarding it. The λ-analogue of falling factorials are defined as follows:

$$(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda). \tag{14}$$

From the definition of $(x)_{l,\lambda}$ in the equation (14), we get

$$\begin{aligned}\left(\frac{x}{\lambda}\right)_l &= \frac{1}{\lambda}\left(\frac{x}{\lambda}-1\right)\left(\frac{x}{\lambda}-2\right)\cdots\left(\frac{x}{\lambda}-l+1\right)\\ &= \frac{x}{\lambda^{l+1}}(x-\lambda)(x-2\lambda)\cdots(x-(l-1)\lambda)\\ &= \lambda^{-l}(x)_{l,\lambda}.\end{aligned} \tag{15}$$

The λ-*analogue of the Stirling numbers of the first kind* are introduced in [16] as follows:

$$(x)_{n,\lambda}=\sum_{l=0}^{n} S_{1,\lambda}(n,l)x^l,\ (n\geq 0). \tag{16}$$

The coefficients $S_{1,\lambda}(n,l)$ on the right side of (16) are called the λ- analogue Stirling numbers of the first kind.

Let us apply the equation (13),(15) and (16) to the equation (12),

$$\begin{aligned}&\frac{\lambda t}{\log(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\ &= \sum_{m=0}^{\infty} C_{m,\lambda,4}\frac{t^m}{m!}\sum_{k=0}^{\infty}\left(\frac{x}{\lambda}\right)_k \frac{\lambda^k\left(\log(1+t)\right)^k}{k!}\\ &= \sum_{m=0}^{\infty} C_{m,\lambda,4}\frac{t^m}{m!}\sum_{k=0}^{\infty}\left(\frac{x}{\lambda}\right)_k \lambda^k\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}\\ &= \sum_{m=0}^{\infty} C_{m,\lambda,4}\frac{t^m}{m!}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda^{-k}(x)_{k,\lambda}\lambda^k S_1(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{l=0}^{k}\binom{n}{m}S_{1,\lambda}(k,l)C_{n-m,\lambda,4}S_1(m,k)x^l\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}S_{1,\lambda}(k,l)S_1(m,k)C_{n-m,\lambda,4}x^l\frac{t^n}{n!}.\end{aligned} \tag{17}$$

The equation (17) give us the following identity.

Theorem 2.1. *For any nonnegative integer $n\in\mathbb{N}^*$,*

$$C_{n,\lambda,4}(x)=\sum_{l=0}^{n}\sum_{m=l}^{n}\sum_{k=l}^{m}\binom{n}{m}S_{1,\lambda}(k,l)S_1(m,k)C_{n-m,\lambda,4}\ x^l.$$

Substituting $\left(e^{\frac{t}{\lambda}}-1\right)$ instead of t in the left side of the equation (12), we get

$$\frac{\lambda\left(e^{\frac{t}{\lambda}}-1\right)}{\log\left(1+\lambda\log\left(1+\left(e^{\frac{t}{\lambda}}-1\right)\right)\right)}\left(1+\lambda\log\left(1+\left(e^{\frac{t}{\lambda}}-1\right)\right)\right)^{\frac{x}{\lambda}} = \frac{\lambda\left(e^{\frac{t}{\lambda}}-1\right)}{\log(1+t)}(1+t)^{\frac{x}{\lambda}} = \lambda\frac{t}{\log(1+t)}(1+t)^{\frac{x}{\lambda}}\frac{\left(e^{\frac{t}{\lambda}}-1\right)}{t}. \tag{18}$$

Since $e^{\frac{t}{\lambda}} = 1+\frac{t}{\lambda}+\frac{1}{2!}\frac{t^2}{\lambda^2}+\cdots$, the last line of (18) becomes

$$\begin{aligned}\lambda\frac{t}{\log(1+t)}(1+t)^{\frac{x}{\lambda}}\frac{\left(e^{\frac{t}{\lambda}}-1\right)}{t} &= \sum_{l=0}^{\infty}C_l(\frac{x}{\lambda})\frac{t^l}{l!}\cdot\frac{\lambda}{t}\sum_{m=1}^{\infty}\frac{1}{\lambda^m}\frac{t^m}{\lambda^m}\\ &= \sum_{l=0}^{\infty}C_l(\frac{x}{\lambda})\frac{t^l}{l!}\cdot\sum_{m=0}^{\infty}\frac{1}{\lambda^m(m+1)}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}C_{n-k}(\frac{x}{\lambda})\frac{1}{\lambda^k(k+1)}\right)\frac{t^n}{n!}.\end{aligned} \tag{19}$$

Replacing t by $\left(e^{\frac{t}{\lambda}}-1\right)$ in the right side of the equation (12), the equation turns as follows:

$$\begin{aligned}\sum_{m=0}^{\infty}C_{m,\lambda,4}(x)\frac{\left(e^{\frac{t}{\lambda}}-1\right)^m}{m!} &= \sum_{m=0}^{\infty}C_{m,\lambda,4}(x)\sum_{n=m}^{\infty}S_2(n,m)\frac{1}{\lambda^n}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}C_{m,\lambda,4}(x)S_2(n,m)\frac{1}{\lambda^n}\right)\frac{t^n}{n!}.\end{aligned} \tag{20}$$

The equations (19) and (20) yield the following Theorem 2.2.

Theorem 2.2. *For any nonnegative integer $n\in\mathbb{N}^*$ and $\lambda\in\mathbb{R}$ with $\lambda>0$,*

$$\sum_{k=0}^{n}\binom{n}{k}C_{n-k}\left(\frac{x}{\lambda}\right)\frac{\lambda^{n-k}}{k+1} = \sum_{m=0}^{n}C_{m,\lambda,4}(x)S_2(n,m). \tag{21}$$

If $t=0$ in the (21), we get

$$\sum_{k=0}^{n}\binom{n}{k}C_{n-k}\frac{\lambda^{n-k}}{k+1} = \sum_{m=0}^{n}C_{m,\lambda,4}S_2(n,m). \tag{22}$$

Assume that λ goes to 0 in the equation (22), then left side of the equation (22) remains only in the case $k = n$. This tells us that the following is true.

$$\frac{1}{k+1} = \sum_{m=0}^{k} C_m S_2(k,m), \tag{23}$$

The n-th harmonic series, denoted by H_n, is very well known and extensively have been studied.

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Let us add k on the both side of the equation (23) from 0 to n, then we get the following Corollary (2.3).

Corollary 2.3. *For any nonnegative integer $n \geq 0$,*

$$\begin{aligned} H_{n+1} &= \sum_{k=0}^{n} \sum_{m=0}^{k} C_m S_2(k,m) \\ &= \sum_{m=0}^{n} \sum_{k=m}^{n} C_m S_2(k,m). \end{aligned} \tag{24}$$

Let us replace t by $\lambda \log(1+t)$ in the definition of the Cauchy polynomials, the equation (1), then the definition becomes

$$\frac{\lambda \log(1+t)}{\log(1+\lambda \log(1+t))}(1+\lambda \log(1+t))^x = \sum_{n=0}^{\infty} C_n(x) \frac{(\lambda \log(1+t))^n}{n!}. \tag{25}$$

From the left side of the equation (25), we obtain

$$\begin{aligned} &\frac{\lambda \log(1+t)}{\log(1+\lambda \log(1+t))}(1+\lambda \log(1+t))^x \\ &= \frac{\log(1+t)}{t} \frac{\lambda t}{\log(1+\lambda \log(1+t))}(1+\lambda \log(1+t))^{\frac{\lambda x}{\lambda}} \\ &= \sum_{l=0}^{\infty} D_l \frac{t^l}{l!} \sum_{m=0}^{\infty} C_{m,\lambda,4}(\lambda x) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} D_{n-m} C_{m,\lambda,4}(\lambda x) \right) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

The other side of the equation (25) becomes

$$\begin{aligned} \sum_{m=0}^{\infty} C_m(x) \frac{(\lambda \log(1+t))^m}{m!} &= \sum_{m=0}^{\infty} C_m(x) \lambda^m \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} C_m(x) \lambda^m S_1(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{27}$$

From (26) and (27), we get the following Theorem 2.4.

Theorem 2.4. *For any nonnegative integer $n \geq 0$ and real $\lambda \in \mathbb{R}$ with $\lambda > 0$,*

$$\sum_{m=0}^{n} \binom{n}{m} D_{n-m} C_{m,\lambda,4}(\lambda x) = \sum_{m=0}^{n} C_m(x) \lambda^m S_1(n,m). \tag{28}$$

Since $C_{m,\lambda,4}(0) = C_{m,\lambda,4}$, $C_m(0) = C_m$ and $\lim_{\lambda \to 0} C_{m,\lambda,4} = C_m$, we get the following interesting identity (2) as λ goes to 0.

$$\sum_{m=0}^{n} \binom{n}{m} D_{n-m} C_m = 0.$$

From the definition of the degenerate Cauchy polynomials of the fourth kind, the equation (12), we obtain

$$\begin{aligned}
&\frac{\lambda t}{\log(1+\lambda \log(1+t))}(1+\lambda \log(1+t))^x \\
&= \sum_{l=0}^{\infty} C_{l,\lambda,4} \frac{t^l}{l!} \sum_{m=0}^{\infty} (x)_m \lambda^m \frac{\log(1+t)^m}{m!} \\
&= \sum_{l=0}^{\infty} C_{l,\lambda,4} \frac{t^l}{l!} \sum_{m=0}^{\infty} (x)_m \lambda^m \sum_{k=m}^{\infty} S_1(k,m) \frac{t^k}{k!} \\
&= \sum_{l=0}^{\infty} C_{l,\lambda,4} \frac{t^l}{l!} \sum_{k=0}^{\infty} \sum_{m=0}^{k} (x)_m \lambda^m S_1(k,m) \frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{n}{k} C_{n-k,\lambda,4} (x)_m \lambda^m S_1(k,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{29}$$

The equation (29) provides an explicit formula for the degenerate Cauchy polynomials of the fourth kind.

Theorem 2.5. *For any nonnegative integer $n \in \mathbb{N}^*$ and real $\lambda \in \mathbb{R}$ with $\lambda > 0$,*

$$C_{n,\lambda,4}(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{n}{k} C_{n-k,\lambda,4} (x)_m \lambda^m S_1(k,m). \tag{30}$$

3. **Degenerate Cauchy numbers of the forth kind**

From the definition of the degenerate Cauchy numbers of the fourth kind, the equation (12),

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n,\lambda,4}\frac{t^n}{n!} &= \frac{\lambda t}{\log(1+\lambda\log(1+t))}\\ &= \frac{\lambda\log(1+t)}{\log(1+\lambda\log(1+t))}\frac{t}{\log(1+t)}\\ &= \sum_{l=0}^{\infty} C_l\lambda^l\frac{(\log(1+t))^l}{l!}\sum_{m=0}^{\infty} C_m\frac{t^m}{m!}\\ &= \sum_{l=0}^{\infty} C_l\lambda^l\sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!}\sum_{m=0}^{\infty} C_m\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n} C_l\lambda^l S_1(n,l)\frac{t^n}{n!}\sum_{m=0}^{\infty} C_m\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{n} C_lC_m\lambda^l S_1(n-m,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Comparing the coefficients on the both sides of (31), we get the following theorem.

Theorem 3.1. *For any nonnegative integer $n \in \mathbb{N}^*$ and real $\lambda \in \mathbb{R}$ with $\lambda > 0$,*

$$C_{n,\lambda,4} = \sum_{m=0}^{n}\sum_{l=0}^{n} C_lC_m\lambda^l S_1(n-m,l)$$

From the equation (31), we get

$$\begin{aligned}\frac{t}{\log(1+t)} &= \sum_{m=0}^{\infty} C_{m,\lambda,4}\frac{t^m}{m!}\frac{\log(1+\lambda\log(1+t))}{\lambda\log(1+t)}\\ &= \sum_{m=0}^{\infty} C_{m,\lambda,4}\frac{t^m}{m!}\sum_{l=0}^{\infty} D_l\lambda^l\frac{(\log(1+t)^l)}{l!}\\ &= \sum_{m=0}^{\infty} C_{m,\lambda,4}\frac{t^m}{m!}\sum_{l=0}^{\infty} D_l\lambda^l\sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!}\\ &= \sum_{m=0}^{\infty} C_{m,\lambda,4}\frac{t^m}{m!}\sum_{n=0}^{\infty}\sum_{l=0}^{n} D_l\lambda^l S_1(n,l)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{m}\sum_{l=0}^{n-m}\binom{n}{m}C_{m,\lambda,4}D_l\lambda^l S_1(n-m,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{32}$$

From the generating function of the Cauchy numbers,

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} C_n\frac{t^n}{n!},$$

and (32), we get the following identity Theorem 3.2.

Theorem 3.2. *For any non negative integer $n \in \mathbb{N}^*$ and positive real $\lambda \in \mathbb{R}$,*

$$C_n = \sum_{m=0}^{m} \sum_{l=0}^{n-m} \binom{n}{m} C_{m,\lambda,4} D_l \lambda^l S_1(n-m,l).$$

We note that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k. \tag{33}$$

From the definition of the degenerate Cauchy numbers of fourth kind (12) and (33), we obtain

$$\begin{aligned}
\lambda t &= \log(1+\lambda \log(1+t)) \sum_{n=0}^{\infty} C_{n,\lambda,4} \frac{t^n}{n!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\lambda^k}{k} (\log(1+t))^k \sum_{n=0}^{\infty} C_{n,\lambda,4} \frac{t^n}{n!} \\
&= \sum_{k=1}^{\infty} (-1)^{k-1}\lambda^k (k-1)! \sum_{l=k}^{\infty} S_1(l,k) \frac{t^l}{l!} \sum_{n=0}^{\infty} C_{n,\lambda,4} \frac{t^n}{n!} \\
&= \sum_{l=1}^{\infty} \sum_{k=1}^{l} (-1)^{k-1}\lambda^k (k-1)! S_1(l,k) \frac{t^l}{l!} \sum_{n=0}^{\infty} C_{n,\lambda,4} \frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \sum_{k=1}^{l} \binom{n}{l} (-1)^{k-1}\lambda^k (k-1)! S_1(l,k) C_{n-l,\lambda,4} \right) \frac{t^n}{n!}.
\end{aligned} \tag{34}$$

The coefficients of t on the right hand side of the equation (34) do not equal zero only in the case $n = 1$. Therefore we get the following.

Theorem 3.3. *For any positive integer n and positive real λ,*

$$\sum_{l=1}^{n} \sum_{k=1}^{l} \binom{n}{l} (-1)^{k-1}\lambda^k (k-1)! S_1(l,k) C_{n-l,\lambda,4} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

By substituting $\frac{1}{\lambda}$ instead of λ in the definition of the generating function of the degenerate Cauchy numbers (2), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n,\frac{1}{\lambda}} \frac{t^n}{n!} &= \frac{\lambda \log(1+\frac{t}{\lambda})}{\log(1+\lambda\log(1+\frac{t}{\lambda})))} \\
&= \frac{t}{\log(1+\lambda\log(1+\frac{t}{\lambda})))} \frac{\log(1+\frac{t}{\lambda})}{\frac{t}{\lambda}} \\
&= \sum_{l=0}^{\infty} \frac{C_{l,\lambda,4}}{\lambda^l} \frac{t^l}{l!} \sum_{m=0}^{\infty} \frac{(-1)^m m!}{m+1} \frac{1}{\lambda^m} \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{C_{l,\lambda,4}}{\lambda^l} \frac{(-1)^{n-l}(n-l)!}{n-l+1} \frac{1}{\lambda^{n-l}} \right) \frac{t^n}{n!}
\end{aligned} \tag{35}$$

From (35), we get an identity between the degenerate Cauchy numbers and the degenerate Cauchy numbers of the fourth kind.

Theorem 3.4. *For any non negative integer n and positive real λ,*

$$C_{n,\frac{1}{\lambda}} = \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^{n-l} C_{l,\lambda,4}(n-l)!}{\lambda^n (n-l+1)}$$

From the second line in the equation (35), we obtain

$$\frac{t}{\log(1+\lambda\log(1+\frac{t}{\lambda})))} = \frac{\frac{t}{\lambda}}{\log(1+\frac{t}{\lambda})} \cdot \sum_{n=0}^{\infty} C_{n,\frac{1}{\lambda}} \frac{t^n}{n!}. \tag{36}$$

The equation (36) yield

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{C_{n,\lambda,4}}{\lambda^n} \frac{t^n}{n!} &= \sum_{l=0}^{\infty} C_l \frac{1}{\lambda^l} \frac{t^l}{l!} \sum_{m=0}^{\infty} C_{m,\frac{1}{\lambda}} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} C_{n-m} \frac{1}{\lambda^{n-m}} C_{m,\frac{1}{\lambda}} \frac{t^n}{n!}. \end{aligned} \tag{37}$$

The equation (37) yields the following identity.

Theorem 3.5. *For any integer $n > 1$ and positive real λ,*

$$C_{n,\lambda,4} = \sum_{m=0}^{n} \binom{n}{m} \lambda^m C_{n-m} C_{m,\frac{1}{\lambda}} \tag{38}$$

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